MATH 4281 Risk Theory–Ruin and Credibility

Intro to the course and start of Module $1 \label{eq:linear}$

January 12, 2021

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Introduction to this course

Start of Module 1: Intro to Aggregate Loss Models Generating Functions and Convolutions

Introduction to this course

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The story so far...

For those who took Math 4280 you essentially studied the following problem:

How to compute
$$\rho[L]$$
? = ES[L]?

Where:

- *ρ* is some (potentially) coherent risk measure e.g. ES/TCE/CVaR, etc...
- *L* is some random variable representing a loss.

Some Questions

Q1: What do you do when L is equal to a sum of smaller RVs?

Q2: How do you introduce time to this model?

Q3: How do I estimate the parameters of the model for *L*...if I don't have a nice heterogeneous sample?

Some Questions

Q1: What do you do when L is equal to a sum of smaller RVs? \Rightarrow Module 1: Aggregate Loss Models

Q2: How do you introduce **time** to this model? \Rightarrow Module 2: Ruin Theory

Q3: How do I estimate the parameters of the model for *L*...if I don't have a nice heterogeneous sample? ⇒ Module 3: Credibility

Start of Module 1: Intro to Aggregate Loss Models

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Models for aggregate losses

A portfolio of insurance contracts or an insurance contract will potentially experience a sequence of losses:

X_1, X_2, X_3, \ldots

We are interested in the aggregate sum S of these losses over a certain period of time.

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Assumptions going forward

- How do losses relate to each other?
 - \hookrightarrow Assume independent X_i 's.
- When do these losses occur?

 $\,\hookrightarrow\,$ Assume no time value of money i.e. short term models

How many losses will occur?

 → if deterministic (n) → individual risk model
 → if random (N) → collective risk model

Definition: The Individual Risk Model

The Individual Risk Model¹

$$S = X_1 + \cdots + X_n = \sum_{i=1}^n X_i,$$

- The random variables X_i, i = 1, 2, ..., n, are assumed to be independent
- BUT they are not assumed to be identically distributed.
- Typically the X_i's have mass at 0 (representing no loss/payment).

Definition: The Collective Risk Model

In the Collective Risk Model, aggregate losses become

$$S = X_1 + \ldots + X_N = \sum_{i=1}^N X_i.$$

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This is a random sum. We make the following assumptions:

- N is the number of claims
- X_i is the amount of the *i*th claim
- the X_i's are iid with
 - CDF F(x)
 - P(D/M)F f(x)
 - Moments exist and are finite!
- the X_i 's and N are mutually independent

How to compute *S*?

We will study methods to get probabilities about S:

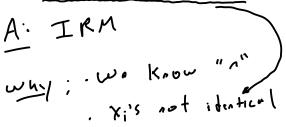
• If possible we will get the true distribution of S via:

- Convolutions
- Method of generating functions

Otherwise we will approximate with the help of the moments of S

Examples of The IRM vs CRM

A group life insurance contract where each employee has a different age, gender, and death benefit².



Examples of The IRM vs CRM

A reinsurance contract that pays when the annual total medical malpractice costs at a certain hospital exceeds a given amount.

Examples of The IRM vs CRM

A dental policy on an individual pays for at most two checkups per year per family member. A single contract covers any size family at the same price.

An example for the collective Risk model

An insurable event has a 10% probability of occurring and when it occurs results in a loss of \$5,000. Market research has indicated that consumers will pay at most \$550 to purchase insurance against this event. How many policies must a company sell in order to have a 95% chance of making money (ignoring expenses)?³

An example for the collective Risk model

$$(5.95 \leq P(SOOD(\leq SSO_n))) = P(C \leq 0.11n))$$

$$= P(C \leq 0.11n))$$

$$= P(C \leq 0.11n))$$

$$= P(C \leq 0.11n))$$

$$= P(C \leq 0.11n))$$

$$(-N(np, np(1-p)))$$

$$= \frac{1}{(J = 0.03)}$$

$$= \sum_{j=1}^{\infty} (0.9j) \leq \frac{1}{J_{TD}} = 0.03$$

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An example for the collective Risk model

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Generating Functions Convolutions

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Generating Functions and Convolutions

Generating Functions Convolutions

Probability Generating functions

Definition (PGF)

Given a discrete RV X. We define the Probability Generating Function(PGF) $p_X(t)$ as:

$$p_X(t) = E\left[t^X\right]$$

i.e:

$$p_X(t) = Pr[X = x_0]t^{x_0} + Pr[X = x_1]t^{x_1} + Pr[X = x_2]t^{x_2} + \dots$$

Properties

- There is a 1-1 relation between a distribution and its PGF.
- If X is an integer-valued random variable, then the PGF is

$$p_X(t) = E\left[t^X\right] = \sum_{n=0}^{\infty} \Pr[X=n]t^n$$

Generating Functions

which is in fact the Taylor series of $p_X(t)$:

$$\Pr[X = n] = \frac{\frac{d^n}{dt^n} p_X(t)|_{t=0}}{n!} \begin{bmatrix} \left(c \cos n \right) \\ \rho_{roo} b c b c h^{ij} \end{pmatrix}$$

• If
$$X_i, i = 1, \dots, n$$
 are independent, then:

$$p_{X_{1}+\cdots+X_{n}}(t)=p_{X_{1}}(t)p_{X_{2}}(t)\cdots p_{X_{n}}(t)$$



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Moment Generating functions

Definition (MGF)

For a continous random variable X we define the Moment Generating Function (MGF) as:

$$M_{X}\left(t
ight)=E\left(e^{tX}
ight)$$

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Properties

- There is a 1-1 relation between a distribution and its MGF.
- Taylor Expansion:

$$M_{X}(t) = 1 + E[X]t + E[X^{2}]\frac{t^{2}}{2} + E[X^{3}]\frac{t^{3}}{6} + \dots + E[X^{k}]\frac{t^{k}}{k!} + \dots$$

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 $M_{X_{1}+\cdots+X_{n}}(t) = M_{X_{1}}(t) M_{X_{2}}(t) \cdots M_{X_{n}}(t)$

$$\bigotimes$$

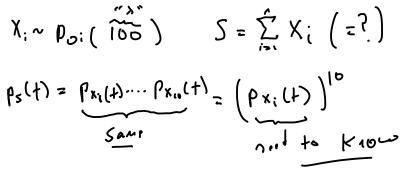
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Why do we care?

- As we said there is a 1-1 relation between a distribution and its MGF or PGF.
- Sometimes, m_S(t) or p_S(t) can be recognised: this is the case for infinitely divisible distributions (Normal, Poisson, Inverse Gaussian, ...) and certain other distributions (Binomial, Negative binomial)
- Otherwise, m_S(t) or p_S(t) can be expanded numerically to get moments and/or probabilities

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Example Consider a portfolio of 10 contracts. The losses X_i 's for these contracts are i.i.d. Poisson RVs with parameter 100. Determine the distribution of S.



Generating Functions Convolutions

$$P_{x_{i}}(t) = \left[\left[\begin{array}{c} t \\ \end{array}\right]^{X} \right] = \sum_{\substack{i=1\\i\neq i}}^{\infty} \left(\begin{array}{c} \lambda^{A} \\ \frac{\lambda^{i}}{\lambda^{i}} \end{array}\right)^{A}$$

$$= \left(\begin{array}{c} -\lambda \\ \sum_{i\neq i}}^{\infty} \\ \frac{\lambda^{i}}{\lambda^{i}} \end{array}\right)^{A} = \left(\begin{array}{c} \lambda^{(i+1)} \\ \frac{\lambda^{i}}{\lambda^{i}} \end{array}\right)^{A}$$

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$$= \left(\begin{array}{c} \lambda^{(i+1)} \\ \frac{\lambda^{i}}{\lambda^{i}} \end{array}\right)^{A}$$

Generating Functions Convolutions

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Generating Functions Convolutions

Example Consider three independent RVs X_1 , X_2 , X_3 . For i = 1, 2, 3, X_i has an exponential distribution and $E[X_i] = 1/i$. Derive the PDF of $S = X_1 + X_2 + X_3$ by recognition of the MGF of S.

$$MGF; E[r^{+Y_i}] = \frac{1}{i^{-+}} \quad \text{for } Ici (E \times mis_r)$$

$$M_{s}(L) = M_{x_{1}}(L) \cdot M_{x_{n}}(L) \cdot M_{x_{n}}(L)$$
$$= \left(\frac{1}{1 \cdot t} \right) \cdot \left(\frac{2}{1 \cdot t} \right) \cdot \left(\frac{3}{3 - t} \right)$$

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$$M_{s}(t) = \left(\frac{1}{1-t}\right) \left(\frac{2}{2-t}\right) \left(\frac{3}{3-t}\right)$$

$$= 3\left(\frac{1}{1-t}\right) - 3\left(\frac{2}{2-t}\right) + \left(\frac{3}{3-t}\right)$$

$$F_{metrone}$$
Not:

$$\frac{10010}{5(x)^{2}} = \frac{1}{3}(x) + h(x) = E[x^{+x}] = \int e^{+x} (j+4) dx = \int f f$$

$$\int f_{x}(x) = -\frac{3}{3}(x^{-x}) - \frac{3}{3}(x^{-2x}) + (\frac{3}{3}e^{-3x})$$

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Generating Functions Convolutions

Convolutions

• The operation of computing the distribution of the sum of two independent random variables is called a convolution. It is denoted by:

$$F_{X+Y} = F_X * F_Y$$

• The result can then be convoluted with the distribution of another random variable:

$$F_{X+Y+Z} = F_Z * F_{X+Y}$$

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• And so on...(as we will see for *n*-fold convolutions)

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Formulas

Continuous case:

• CDF:
$$F_{X+Y}(s) = \int_{-\infty}^{\infty} F_Y(s-x) f_X(x) dx$$

• PDF: $f_{X+Y}(s) = \underbrace{\int_{-\infty}^{\infty} f_Y(s-x) f_X(x)}_{p \text{ if of } f_Y(s)} dx = \int_{\gamma}^{\gamma} \bigoplus \int_{x}^{\gamma} \int_{x}^{y} \int_{x}^{y} \int_{y}^{y} \int_{x}^{y} \int_{x}^{y} \int_{y}^{y} \int_$

Discrete case:

• CDF:
$$F_{X+Y}(s) = \sum_{x} F_{Y}(s-x) f_{X}(x)$$

• PMF:
$$f_{X+Y}(s) = \sum_{x} f_{Y}(s-x) f_{X}(x)$$

Generating Functions Convolutions

n-fold convolutions

For i.i.d. continuous random variables X_i with a common CDF $F_X(x)$, the *n*-fold convolution of $F_X(x)$ is denoted by $F_X^{*n}(x)$:

$$F_X^{*k}(x) = \int_{-\infty}^{\infty} F_X^{*(k-1)}(x-y) f_X(y) dy$$

=
$$\int_0^x F_X^{*(k-1)}(x-y) f_X(y) dy \quad \text{if positive support}$$

for k = 1, 2, ... where:

$$F_X^{*0}(x) = egin{cases} 0, & x < 0 \ 1, & x \ge 0 \end{cases}$$

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n-fold convolutions (Cont.)

Continuous case PDF for k = 1, 2, ...:

$$f_X^{*k}(x) = \int_{-\infty}^{\infty} f_X^{*(k-1)}(x-y) f_X(y) dy$$

=
$$\int_0^x f_X^{*(k-1)}(x-y) f_X(y) dy \quad \text{if positive support}$$

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n-fold convolutions (Cont.)

Discrete case:

• CDF:

$$F_X^{*k}(x) = \sum_{y=0}^{x} F_X^{*(k-1)}(x-y) f_X(y)$$
 for $x = 0, 1, ..., k = 2, 3, ...$

• PMF:

$$f_X^{*k}(x) = \sum_{y=0}^{x} f_X^{*(k-1)}(x-y) f_X(y)$$
 for $x = 0, 1, ..., k = 2, 3, ...$

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Exercise: Show the convolution gives the distribution for the sum.

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