MATH 4281 Risk Theory–Ruin and Credibility

Module 1 (cont.)

January 26, 2021

- Last week we looked at some more applied insurance problems with the IRM.
- Today we will return to the CRM.
- The mathematics of the CRM are slightly more complicated.
- This will also flow nicely into Module 2.

The Collective Risk Model

Different ways of separating frequency and severity

The IRM - deterministic n

- main focus on the claims of individual policies (whose number is a priori known)
- $\longrightarrow$ Individual Risk Model

The CRM - random N

- main focus on claims of a whole portfolio (whose number is a priori unknown)
- $\longrightarrow$ Collective Risk Model

Definition

In the Collective Risk Model, aggregate losses become

$$S = X_1 + \ldots + X_N = \sum_{i=1}^{N} X_i.$$

This is a random sum. We make the following assumptions:

- N is the number of claims
- X_i is the amount of the ith claim
- the X_i's are i.i.d with CDF $F(x)$ and PDF/PMF $f(x)$
- Moments $E[X^k] = \mu'_k$ (particularly, $E[X] = \mu'$)[1]
- the X_i's and N are mutually independent

[1] The primes so we can distiguish them from the moments of S i.e. $E[S] = \mu$.

Moments of S

We have

$$E[S] = E\left[E[S|N]\right] = E\left[NE[X]\right] = E[N]\mu,$$

and

$$\begin{aligned} Var(S) &= E\left[Var(S|N)\right] + Var\left(E[S|N]\right) \\ &= E\left[NVar(X)\right] + Var(\mu N) \\ &= E[N]Var(X) + \mu^2 Var(N) \\ &= E[N](\mu_2' - \mu^2) + \mu^2 Var(N) \\ &= E[N]\mu_2' + \mu^2 \left\{Var(N) - E[N]\right\}. \end{aligned}$$

MGF of S as a function of $M_X(t)$ and $M_N(t)$

$$M_S(t) = E[e^{tS}]$$

$$= E[\, E[e^{t(X_1 + \cdots + X_n)} \mid N = n]\,]$$

$$= E[(M_X(t))^N]$$

$$= E[e^{(\ln(M_X(t))N}]$$

$$= M_N(\ln(M_X(t)))$$

PGF?

$$P_S(t) = E[\ t^S\ ]$$

$$= \sum_{k=0}^{\infty} P(S=k)\, t^k$$

$$= \sum_{k=0}^{\infty} \left[\sum_{n=0}^{\infty} P(\underline{S=k} \mid N=n) \cdot P(N=n) \right] t^k$$

Rearrange . . .

$$= \sum_{n=0}^{\infty} P(N=n) \underbrace{\left[\sum_{k=0}^{\infty} P\left(\sum_{i=1}^{n} X_i = k \mid N=n\right) t^k \right]}_{(P_X(t))^n}$$

$$= P_N(P_X(t))$$

So in conclusion we have:

- MGF: $M_N(\ln M_X(t))$

- PGF: $P_S(t) = P_N[P_X(t)]$

Example

Assume that N is geometric with probability of success p. Find $M_S(t)$ in terms of $M_X(t)$.

Geo. pmf: $p(1-p)^{n-1}$

$$M_S(t) = M_N(\ln(M_X(t))) \Rightarrow M_N = ?$$

$$M_N(t) = \mathbb{E}[e^{tN}] = \sum_{n=1}^{\infty} P(N=n)\, e^{tn} = \frac{p}{(1-p)} \sum_{n=1}^{\infty} \left[(1-p)e^t\right]^n$$

$$= \left(\frac{p}{1-p}\right) \frac{1}{1-(1-p)e^t}$$

$$\boxed{M_S(t) = \frac{p}{1-p} \, \frac{1}{1-(1-p)M_X(t)}}$$

Popular options for the distribution of N

- Poisson(λ)
  - $E[N] = Var(N) = \lambda$
  - S is a compound Poisson with parameters $(\lambda, F_X(x))$
- Negative Binomial(r, β)
  - $E[N] < Var(N)$
  - S is a compound Negative Binomial with parameters $(r, \beta, F_X(x))$
- Binomial(m, q)
  - $E[N] > Var(N)$
  - S is a compound Binomial with parameters $(m, q, F_X(x))$
  - least popular

Most Important Example!

If N is Poisson with intensity λ, then $S = \sum_{i=1}^{N} X_i$ follows a Compound Poisson Distribution.

1. MGF:

$$M_S(t) = ?$$

2. PGF:

$$P_S(t) = exp\{\lambda(P_X(t) - 1)\}$$

MGF of a compound Poisson

Really comes down to taking the MGF of a Poisson distribution.

$$M_S(t) = M_N(\ln(M_X(t))) \; ; \quad M_N(t) = ?$$

$$E[e^{tN}] = \sum_{k=0}^{\infty} \underbrace{\left(\frac{e^{-\lambda}\lambda^k}{k!}\right)}_{= P(N=k)} e^{tk} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^t \lambda)^k}{k!}$$

$$= e^{-\lambda}\left(e^{\lambda e^t}\right) = e^{\lambda(e^t - 1)}$$

$$M_S(t) = e^{\lambda(M_X(t) - 1)}$$

Quick Aside on PGFs, MGFs, etc

- Why is MGF/PGF of compound Poisson so similar?
- Well superficially:

$$E[t^X] = E[e^{\log(t)X}], t > 0$$

- Use PGFs for discrete distributions $\rightarrow$ gives a power series and the results therin (e.g. Abel's theorem).
- Use MGFs for continuous $\rightarrow$ gives an integral transform and results from Laplace/Fourier analysis can be used.
- But as long as everything converges nicely- nothing stopping you from taking MGFs of discrete and vice versa. May not be useful however.

Cumulants

- Define the k-th cumulant of the random variable Y:

$$\kappa_k = \frac{d^k}{dt^k}\kappa_Y(t)\bigg|_{t=0} = \frac{d^k}{dt^k}\ln(M_Y(t))\bigg|_{t=0}$$

- Similar to Moments[2] but with the key difference that cumulative are related to *Central Moments*!
- For example:
    - Mean: κ_1
    - Variance: κ_2
    - Skewness ($\gamma_1(Y)$): $\frac{\kappa_3}{\kappa_2^{3/2}}$
    - Kurtosis ($\gamma_2(Y)$): $\frac{\kappa_4}{\kappa_2^2}$

[2] Related to our previous discussion there is also a Cumulant Generating Function $K_Y(t) = \log(M_Y(t))$

Cumulants of a Compound Poisson

In the case of a compound Poisson random variable we have

$$\kappa_k = \frac{d^k}{dt^k}\lambda(M_X(t)-1)\bigg|_{t=0} = \lambda\,\frac{d^k}{dt^k}M_X(t)\bigg|_{t=0} = \lambda\mu'_k.$$

Thus

$$\begin{aligned} E[S] &= \lambda\mu \quad \text{and} \quad Var(S) = \lambda\mu'_2 \\ \gamma_1(S) &= \frac{\lambda\mu'_3}{(\lambda\mu'_2)^{\frac{3}{2}}} = \frac{\mu'_3}{\sqrt{\lambda}(\mu'_2)^{3/2}} \\ \gamma_2(S) &= \frac{\lambda\mu'_4}{(\lambda\mu'_2)^2} = \frac{\mu'_4}{\lambda(\mu'_2)^2} \end{aligned}$$

A *Very* Important Theorem

The sum of m independent compound Poisson $(\lambda_i, F_i(x))$ random variables, i.e.,

$$S = \sum_{i=1}^{m} S_i, \quad S_i \sim (\lambda_i, F_i(x))$$

is a compound Poisson random variable again with parameters

$$\lambda = \sum_{i=1}^{m} \lambda_i \quad \text{and} \quad F(x) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} F_i(x).$$

So what?

- Independent portfolios of losses can be easily aggregated.
- Total claims paid over m years is compound Poisson, even if the severity and frequency of losses vary across years.
- The time value of money can be approximated by a change of scale on F_i for each year.

Proof

$S_i \sim$ Cmpd Poisson $\bar{w}$ $M_{S_i}(t) = \exp\left\{ \lambda_i (M_{X_i}(t) - 1) \right\}$

· Want $M_S(t)$ where $S = \sum S_i$

· Note S_i's are ind. by assumption

$$M_S(t) = \prod_{i=1}^{m} M_{S_i}(t) = \exp\left\{ \sum_{i=1}^{m} \lambda_i (M_{X_i}(t) - 1) \right\}$$

Say $\lambda = \sum \lambda_i$

$$= \exp\left\{ \lambda \left(\sum_{i=1}^{m} \frac{\lambda_i}{\lambda} M_{X_i}(t) - 1 \right) \right\}$$

Proof

$$M_S(t) = \exp\left\{ \lambda \left(\sum_{i=1}^{m} \frac{\lambda_i}{\lambda} M_X(t) - 1 \right) \right\}$$

Need this to be a MGF

Indeed $\sum_{i=1}^{m} \frac{\lambda_i}{\lambda} M_{X_i}(t)$ is the MGF of

$$F(x) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} F_i(x)$$

Why? $E[e^{tX}] = \sum \int \frac{\lambda_i}{\lambda} e^{tx} f_i(x)\,dx$

Distribution of S

It is possible to get a fairly general expression for the CDF of S by conditioning on the number of claims:

$$F_S(x) = \sum_{n=0}^{\infty} \Pr[S \leq x | N = n] \Pr[N = n] = \sum_{n=0}^{\infty} F_X^{*n}(x) p_n,$$

where $F_X^{*n}(x)$ is the n-fold convolution of $F_X(x)$.
Note that

- N will always be discrete, so this works for any type of RV X (continuous, discrete or mixed)
- however, the type of S will depend on the type of X

Next Class: end of Module 1

- Next class we will discuss various ways to approximate $F_S(x)$.

- We will broadly do this is 2 ways:
  1. Recursion algorithms
  2. The Central Limit Theorem

- I will also start to post a bank of study questions this week for your Module 1 test in February.