MATH 4281 Risk Theory–Ruin and Credibility

Start Module 2: Ruin Theory

Feb 2, 2021

Recap and Motivation

The Story so Far

- As it stands now- the models we have studied have assumed a short time frame.
- We have tried to model aggregate claims over say a week/month/year.
- Assumed no Time Value of Money or rigorous model for Premiums.

Some Questions

Q1 What happens if we can't pay all the claims?

Q2 How do we set premiums to guarantee that we can?

Q3 How does Time factor in to this?

These are the questions that we will explore in this module

Stochastic Processes

What is a Stochastic Process

- Stochastic - from the Greek for "to aim" or "to guess". Generally adjective denoting "randomness" e.g:
  - stochastic process (mathematics)
  - stochastic resonance (biology)
  - newsworthy "stochastic terrorism" (social sciences)
  - etc...

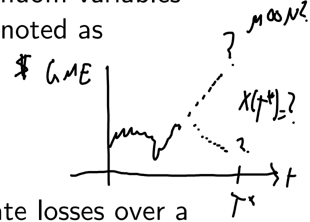
- Process - Latin for "progression"

- Stochastic Process - stands to reason this is some progression of random events

What is a Stochastic Process

- A *stochastic process* is any collection of random variables $X(t)$, $t \in T$. This stochastic process is denoted as

$$\{X(t), t \in T\}.$$



- We are interested in modelling the aggregate losses over a given period of time, not necessarily at one point!

- For example: the aggregate loss <u>*process*</u> denoted by $\{S(t), t \geq 0\}$, where $S(t)$ is the aggregate loss at time t.

Independent Increments

A stochastic process $\{X(t), t \geq 0\}$ has *independent increments* if:

- For all $t_0 < t_1 < t_2 < \cdots < t_n$ the following RVs[1] are independent:

$$\underbrace{X(t_1) - X(t_0)}_{Y_1}, \underbrace{X(t_2) - X(t_1)}_{Y_2}, \ldots, X(t_n) - X(t_{n-1})$$

Y_2 ind Y_1

- That is, future increases are independent of the past.

[1] RV = Random Variable

Stationary Increments

A stochastic process $\{X(t), t \geq 0\}$ has *stationary increments* if:

- for all choices of t_1, t_2 and $\tau > 0$:

$$X(t_2 + \tau) - X(t_1 + \tau) \overset{d}{=} X(t_2) - X(t_1)$$

- Equivalently for $s < t$

$$X(t) - X(s) \overset{d}{=} X(t - s)$$

Counting process

- A stochastic process $\{N(t), t \geq 0\}$ is a *counting process* if it represents the number of events that occur up to time t.
- Q: What is the significance of counting processes?
- A: We will use them to model the number of claims recived during a particular time.

Counting process

A counting process $\{N(t), t \geq 0\}$ must satisfy:

1. $N(t) \geq 0$.
2. $N(t)$ is integer-valued.
3. $N(s) \leq N(t)$ for any $s < t$, i.e. it must be non-decreasing.
4. For $s < t$, $N(t) - N(s)$ is the number of events that have occurred in the interval $(s, t]$.

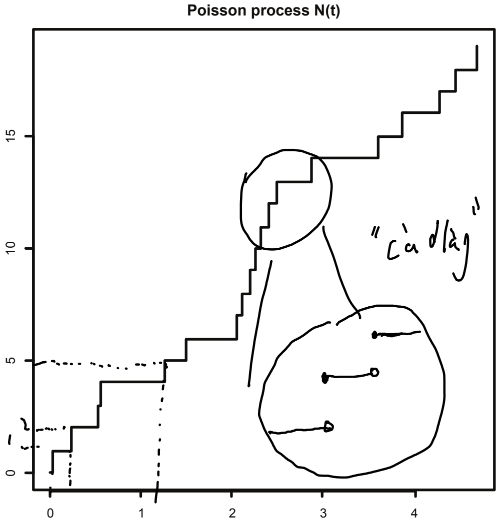
Review

A counting process $\{N(t), t \geq 0\}$ is a *Poisson process* with rate λ, for $\lambda > 0$, if:

1. $N(0) = 0$;
2. it has independent increments; and
3. the number of events in any interval of length t has a Poisson distribution with mean λt. That is, for all $s, t \geq 0, n = 0, 1, ...$

$$\Pr[N(t+s) - N(s) = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

A path (realization) of the Poisson process



- Counting process
- Step function
- What is the arrival time?

A Noteworthy Characterization

Theorem:

- Consider the time from the $(i-1)$th and ith jump W_i.
- That is $t = W_1 + \ldots + W_{N(t)}$
- Then $N(t+h) - N(t) \sim \text{Poi}(\lambda h)$ iff $W_i \sim \text{Exp}(1/\lambda)$

Proof:

$$(W_i \le \tau) \equiv (N(t_i+\tau) - N(t_i) > 0)$$

$$P(W_i \le \tau) = P(N(t_i+\tau) - N(t_i) > 0)$$

compliment

$$P(W_i \le \tau) = 1 - P(N(t_i-\tau) - N(t_i) = 0)$$

$$= 1 - \frac{(\lambda\tau)^0}{0!} e^{-(\lambda\tau)} = \underbrace{1 - e^{-\lambda\tau}}_{\text{Exp}(1/\lambda)}$$

A Noteworthy Characterization

- Recall also that there is something special about the .
distribution!
- Exponential waiting times are Memoryless

E.g:

$$P(W > t+s \mid W > t) = P(W > s) \quad (*)$$

$$\frac{P(W > t+s)}{P(W > t)} = P(W > s) \Rightarrow \underbrace{S(t+s) = S(t)S(s)}$$

Exponential Survival Function is only soln to the functional equation above

$\boxed{* S \to \text{survival fn}}$

Zooming in on the process

Explain why the following are true:

$$
\begin{aligned}
P\left[N(t+dt)-N(t)=1 \mid N(s), 0 \le s \le t\right] &= \lambda dt + o(dt^2) \\
P\left[N(t+dt)-N(t)=0 \mid N(s), 0 \le s \le t\right] &= 1-\lambda dt + o(dt^2) \\
P\left[N(t+dt)-N(t)\ge 2 \mid N(s), 0 \le s \le t\right] &= o(dt^2)
\end{aligned}
$$

i) $P(N(t+dt)-N(t)=1) = \dfrac{e^{-(\lambda dt)}(\lambda dt)^1}{1!}$

$= \lambda dt[1-(-\lambda t)+O(dt^2)]$

$= \lambda dt + \underbrace{\lambda^2 dt^2 + \cdots}_{O(dt^2)}$

N.B

$f(x) \in O(g(x))$ if

$\lim_{x\to 0} \dfrac{f(x)}{g(x)} = 1$

ii)

$$P(N(t+dt)-N(t)=0)=\frac{(\lambda dt)^0}{0!}e^{-(\lambda dt)}$$

$$=\left[1-\lambda dt+o(dt^2)\right]$$

iii) $P(N(t+dt)-N(t)\geq 2)$

$=1-P(N(t+dt)-N(t)<2)$

$=$ ..... ?

(Exercise)

Properties of the Poisson process: Summary

If $\{N(t), t \geq 0\}$ is a *Poisson process* with rate λ, for $\lambda > 0$, then

1. $N(0) = 0$;
2. it has independent and stationary increments;
3. It can never have more than 1 jump at a time! That is:

$$\Pr[N(t+h) - N(t) = 0] = e^{-\lambda h} = 1 - \lambda h + o(h)$$

$$\Pr[N(t+h) - N(t) = 1] = \lambda h e^{-\lambda h} = \lambda h + o(h)$$

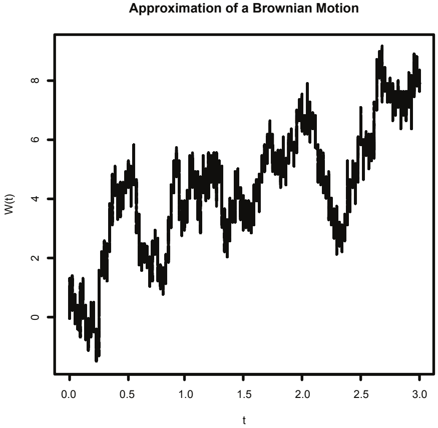
and

$$\Pr[N(t+h) - N(t) \geq 2] = o(h)$$

4. The time between two consecutive jumps follows the Exponential(λ) distribution.

Prev. slides

Brownian motion as the limit of a shifted Poisson process



$(\mu = 2, \quad \sigma = 5, \text{ and } \tau = 0.02)$

Consider the following shifted Poisson process:

$$W(t) = \tau N(t) - ct.$$

Increments have moments

$$E[W(t+h) - W(t)] = (\tau\lambda - c)h \equiv \mu h,$$

$$Var(W(t+h) - W(t)) = (\tau^2\lambda)h \equiv \sigma^2 h$$

When $\tau \to 0$ for fixed μ and σ^2, $\{W(t)\}$ becomes a Brownian motion with parameters μ and σ^2.

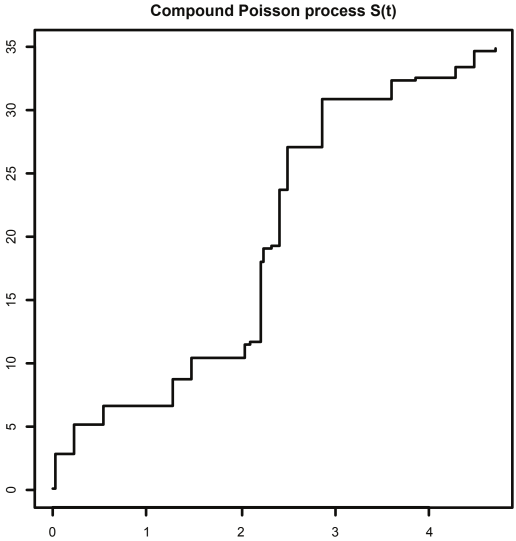
We define a Compound Poisson process $\{S(t), t \geq 0\}$ like so:

$$S(t) = \sum_{i=1}^{N(t)} X_i.$$

Where:

- $\{N(t)\}$ is a Poisson process with parameter λ
- $\{X_i\}$ are iid $\sim P(x)$

A path (realization) of the compound Poisson process



- Now step i has height X_i instead of 1.
- Increments $S(t+h) - S(t) \sim$ Compound Poisson$(\lambda h, P(x))$

Mean and Variance of the compound Poisson process:

$$E[S(t)] = \lambda t E[X], \quad \mathrm{Var}[S(t)] = \lambda t E[X^2].$$

Law of total prob / Tower Property

Same but law of total Var (Exercise)

$$E\left[E\left[S(t) \mid N(t) = n \right] \right]$$

$$= E\left[E\left[\sum^{N} X_i \mid N(t) = n \right] \right]$$

$$= E[N(t)]\, E[X_i]$$

$$= (\lambda t) E[X_i]$$

The MGF of the compound Poisson process:

$$M_{S(t)}(z) = \exp\{\lambda t[M_X(z) - 1]\}.$$

Pf:

$$E\left[e^{zS(t)}\right]$$

$$= E\left[E\left[e^{z\sum^{N_t} X_i} \mid N(t)=n\right]\right]$$

$$= E\left[\left(M_{X_i}(z)\right)^{N(t)}\right]$$

$$= E\left[e^{N(t)\ln(M_{X_i}(z))}\right] = M_{N(t)}\left(\ln(M_{X_i}(z)\right)$$