MATH 4281 Risk Theory–Ruin and Credibility

Module 2: Ruin Theory (cont.)

Feb 23, 2021

.

1. Post Reading Week/test #1 Review

2. The Lundberg Inequality

3. Proving the Cramér-Lundberg Inequality

Post Reading Week/test #1 Review

Recall the following

Since it's been since the 4th we will review what we covered of Ruin Theory so far. Recall we covered:

- Stochastic processes and their properties (independent \stationary increments, etc...) .
- Counting processes, specifically Poisson processes.
- Compound Poisson processes.
- Leading to the Cramér-Lundberg process:

i.e.

$$U(t) = \underbrace{u_0 + ct}_{\text{Revenue}} - \underbrace{\sum_{i=1}^{N(t)} X_i}_{\text{Losses}}$$

The probability of ruin

- Recall the Cramér-Lundberg model:

$$U(t) = u_0 + ct - \sum_{i=1}^{N(t)} X_i$$

- The time to ruin T is defined as

$$T = \inf\{t \geq 0 | U(t) < 0\}.$$

- The probability that the company would be ruined by time t is denoted by

$$\psi(u_0, t) = \Pr[T < t].$$

Avoiding Ultimate Ruin

- Finally, the probability of ultimate ruin is

$$\psi(u_0) = \Pr(T < \infty) = \lim_{t \to \infty} \psi(u_0, t) \geq \psi(u, t).$$

- The Net Profit Condition (NPC):

$$c \leq \lambda \mathbb{E}[X_i] \Rightarrow \psi(u_0) = 1$$

- To ensure the NPC holds we add our "safety loading" :

$$c = (1 + \theta)\lambda \mathbb{E}[X]$$

Recall we introduced an approximation

We can approximate ψ easy via The Lundberg Inequality:

$$\psi(u) \leq e^{-Ru}$$

Where R (the adjustment coefficient) solves the equation[1]

$$e^{rpt} = \mathbb{E}[e^{rS_t}]$$

Today we will discuss this in more detail.

[1] Where $S_t = \sum_{i=1}^{N(t)} X_i$

The Lundberg Inequality

Avoiding Ultimate Ruin

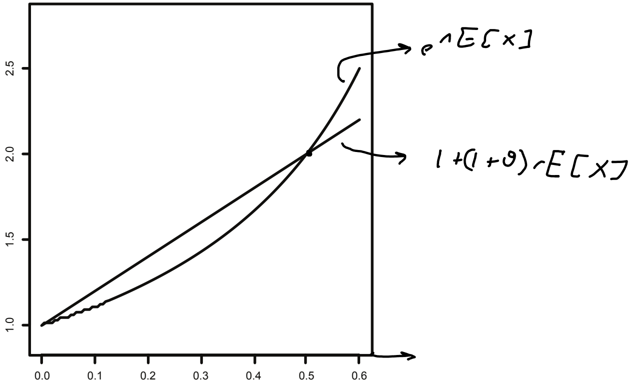
In the Cramér-Lundberg model, consider the excess of losses over premiums over the interval $[0, t]$: $S(t) - ct$. We define the adjustment coefficient R as the first positive solution of the following equation in r:

$$M_{S(t)-ct}(r) = E\left[e^{r(S(t)-ct)}\right] = e^{-rct}e^{\lambda t[M_X(r)-1]} = 1,$$

Recall $c = (1+\theta)\lambda E[X]$. So, the adjustment coefficient R is the first positive of the following equation:

$$1 + (1+\theta)rE[X] = M_X(r)$$

Does such an R exist?



- Recall that Jesen's inequalty gives $1+(1+\theta)rE[X] = \boxed{M_X(r) \geq e^{rE[X]}}$
- How else could this fail?

The Theorem

1. Let $R > 0$ be the adjustment coefficient. If $\{U(t)\}$ is a Cramér-Lundberg process with $\theta > 0$, then for $u \geq 0$

$$\psi(u) = \frac{e^{-Ru}}{E\left[e^{-RU(T)} \middle| T < \infty\right]}.$$

2. Since $U(T) < 0$, we have then (Lundberg's exponential upper bound)

$$\psi(u) < e^{-Ru}.$$

An Example

Assume $X \sim \exp(\beta)$ (the mean is $1/\beta$). Find R and $\psi(u)$.

i) $\underline{R}$

$$e^{rct} = E[e^{rS_t}]$$

$$e^{rct} = \exp\{\lambda t(M_X(r) - 1)\}$$

$$\Rightarrow \quad rc = \lambda(M_X(r) - 1)$$

$$1 + r(c/\lambda) = \frac{1}{1 - r/\beta}$$

$$[1 + r(c/\lambda)] \cdot [1 - r/\beta] = 1$$

$$\cancel{1} + (c/\lambda - 1/\beta)\cancel{r} - (c/\lambda)(1/\beta)r^{\cancel{2}} = \cancel{1}$$

$$\Rightarrow r = \beta - \lambda/c > 0$$

Recall

$$M_X(r) = \frac{\beta}{\beta - r} \quad (MGF)$$

$\underline{\underline{NPC}}$

$c > \lambda E[X]$

$c > \lambda/\beta$

$\lambda/c < \beta$

An Example

So then $R = \beta - \lambda/c$

$$= \beta - \frac{\lambda}{(1+\theta)\lambda(1/\beta)}$$

$$= \beta \frac{\theta}{1+\theta}$$

$$\Rightarrow \boxed{\psi(u) \le \exp\left\{-\beta \frac{\theta}{1+\theta} u\right\}}$$

(N.B. $\psi(u) = \frac{1}{1+\theta} \exp\left\{-\beta \frac{\theta}{1+\theta} u\right\}$ Almost never get a "nice" formula like this!)

Proving the Cramér-Lundberg Inequality

Proving our theorems

To start show that $\{e^{-RU(t)}\}$ is a martingale[2]

adj. coe.

X_t is a s.p. then $E[X_t \mid X_{t-1}] = X_{t-1}$

$$E\left[e^{-RU(t)} \mid e^{-RU(s)}\right] \quad s<t$$

$$= E\left[\exp\left\{-R\left(u_0 + c(\overbrace{t-s}+s) + \sum_{i=1}^{N(s)} X_i + \sum_{i=N(s)}^{N(t)} X_i\right\}\right) \middle| e^{-RU(s)}\right]$$

$$= E\left[\underbrace{\exp\left\{-R\left(u_0 + cs + \sum_{i=1}^{N(s)} X_i\right\}\right)}_{=e^{-RU(s)}} \exp\left\{-R\underbrace{\left(c(t-s) + \sum_{i=N(s)}^{N(t)} X_i\right)}_{\stackrel{d}{=} c(t-s)+\sum_{i=1}^{N(t-s)} X_i}\right\} \middle| e^{-RU(s)}\right]$$

$$\Longrightarrow E\left[\exp\left\{-R\left(c(t-s) - S_{t-s}\right)\right\}\right] = 1$$

Hence the definition of "R"!

[2]i.e. $\boxed{E\left[e^{-RU(t)} \mid e^{-RU(s)}\right] = e^{-RU(s)}}$ for $s<t$ or $E\left[e^{-RU(t)}\right] = e^{-Ru}$ for all t

A *very very* useful theorem

(Given we don't have all the machinery we need at this point- we will define a *stopping time* as a random time dependent on another stochastic process exhibiting some behaviour)

Theorem (Optimal Stopping Theorem)

Given a bounded stopping time T, i.e. $T \leq t_0 < \infty$ for a martingale[a] M_t *them:*

$$M_0 = E[M_T]$$

[a] For those who know we must also impose right continuity

An example: Gambler's Ruin

A gambler enters a casino with n dollars and plays a game with a win probability p. He gains \$1 for every win and losses \$1 for every loss. He leaves when he wins N or looses everything. What is the probability he leaves ruined?

X_t – gambler's wealth

$f(X_t) = \lambda^{X_t}$ where $\lambda = \frac{1-p}{p}$

$$\Rightarrow E[\,f(X_{t+1}) \mid \mathcal{F}(X_t)\,] = p\,\lambda^{(X_t+1)} + (1-p)\,\lambda^{(X_t-1)} = \lambda^{X_t} = f(X_t)$$

Hence $f(X_t)$ is a Martingale!

An example: Gambler's Ruin

$$\tau = \inf_{t \geq 0} \{ X_t = 0 \text{ or } N \} \rightarrow \text{stopping time}$$

by $\underline{\underline{\text{O.S.T}}}$

$$\underbrace{E[f(X_0)]}_{f(n)} = E[f(X_\tau)] = P(X_\tau = N) f(N) + (\overbrace{1 - P(X_\tau = N)}^{P(X_\tau = 0)}) f(0)$$

$$\Rightarrow \boxed{P(X_\tau = N) = \frac{1 - f(n)}{1 - f(N)} = \frac{1 - \lambda^n}{1 - \lambda^N}}$$

Proof of the Cramér-Lundberg Inequality

- So we can do something similar with $e^{-RU(s)}$

Now we are interested in $T = \inf_{t \geq 0} \{ U_t < 0 \}$

$\boxed{\text{Problem} \quad P(T = \infty) > 0 \quad \text{i.e. unbounded}}$

take $t_0 < \infty$; $\tau = T \wedge t_0 = \min\{T, t_0\}$ is bounded

Apply O.S.T;

$$e^{-R(U(0))} = E\left[e^{-RU(\tau)} \right]$$

$$e^{-Ru} = E\left[e^{-RU(\tau)} \mid T < t_0 \right] P(T < t_0)$$

$$+ E\left[e^{-RU(\tau)} \mid T \geq t_0 \right] P(T \geq t_0)$$

Proof of the Cramér-Lundberg Inequality

$$e^{-Ru} = E\left[e^{-RU_\tau} \mid T < t_0\right] \overbrace{P(T < t_0)}^{\psi(u)} + \underbrace{E\left[e^{-RU_\tau} \mid T \geq t_0\right] P(T \geq t_0)}$$

$$E\left[e^{-RU_\tau} \mid T \geq t_0\right] = E\left[e^{-RU(t)} \mathbb{1}_{\{T \geq t_0\}}\right]$$

$$= E\left[e^{-RU(t)} \mathbb{1}_{\{U(t) \geq 0\}}\right]$$

since $0 \leq e^{-RU(t)} \mathbb{1}_{\{U(t) \geq 0\}} \leq 1$

$$\Rightarrow e^{-RU(t)} \mathbb{1}_{\{U(t) \geq 0\}} \longrightarrow 0 \quad \text{as} \quad t_0 \longrightarrow \infty$$