.

MATH 4281 Risk Theory–Ruin and Credibility

Module 2: Ruin Theory (cont.)

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Reinsurance Review

Reinsurance definitions review

- Proportional
  - quota share: the proportion is the same for all risks
  - surplus: the proportion can vary from risk to risk

- Non-Proportional
  - (individual) excess of loss: on each individual loss (X_i)
  - stop loss: on the aggregate loss (S)

$\theta = 0$

- Cheap (reinsurance premium is the expected value), or non cheap (reinsurance premium is loaded)

$\theta > 0$

Reinsurance definitions review

- In proportional reinsurance the retained proportion α defines who pays what:
  - the insurer pays $Y = \alpha X$
  - the reinsurer pays $Z = (1 - \alpha)X$
- In (individual) excess of loss reinsurance for each individual loss X, the reinsurer pays the excess over a retention (excess point) d
  - the insurer pays $Y = \min(X, d)$
  - the reinsurer pays $Z = (X - d)_+$
- These are the examples we will study today.

Applying Ruin Theory to Reinsurance

Why would Ruin theory help?

- Ruin theory gives us a model to study *options* about reinsurance
- Note that even if $\psi(u)$ can't be calculated, we can still play with the adjustment coefficient R and have qualitative results about $\psi(u)$.
- We can adjust the adjustment coefficient (hence its name..) to meet a goal, such as
  - maximize $R \Leftrightarrow$ minimize $\psi(u)$
  - find the cheapest reinsurance such that $\psi(u)$ is inferior to some level

Assumptions

- Let $0 \leq h(x) \leq x$ be the amount paid by the reinsurer for a claim with amount x i.e:
  - $h(X) = (1-\alpha)X$ for proportional reinsurance.
  - $h(X) = (X-d)_+$ for excess of loss reinsurance.

$= \max((X-d), 0)$

- Reinsurance is non cheap and that the loading on reinsurance premiums is $\xi > \theta > 0$. So the reinsurance premium say c_h is:

Recall

$c = (1+\theta)\lambda E[X]$

$$c_h = (1+\xi)\lambda E[h(X)]$$

Assumptions

- With reinsurance, the Cramér-Lundberg process becomes

$$U(t) = u + \underline{(c - c_h)}t - \sum_{i=1}^{N(t)} (X_i - h(X_i)).$$

Lower Revenue → In exchange for coverage

- With reinsurance, the adjustment coefficient, R_h, is then the non-trivial solution to

$$\lambda \left[\underbrace{M_{X-h(X)}}_{\text{"X"}}(r) - 1 \right] = (\underbrace{c - c_h}_{\text{"c"}})r.$$

Equivalently,

$$\lambda + (c - c_h)\, r = \lambda \boxed{\int_0^{\infty} e^{r[x-h(x)]} p(x)\, dx.} \rightarrow M_{X-h(X)}$$

Example 1: Proportional Reinsurance

~~Suppose claims form a compound Poisson process, with $\lambda = 1$ and $p(x) = 1$ for $0 < x < 1$. Further premiums are received continuously at the rate of $c = 1$. Find the adjustment coefficient if proportional reinsurance is purchased with $\alpha = 0.5$ and with reinsurance loading equal to $\xi = 100\%$.~~

Ooops; Quiz question :)

Real Q: • $X_i \sim \exp(1)$; $\theta = 0.25$, $\xi = 0.4$

• Prop. Re. w/ α retention level

• What α maximizes R_U?

Solution

For R_h solve; $\lambda\left[M_{X-h(X)}(r)-1\right]=(c-c_h)r$

(1) $M_{X-h(X)}(r)=?$

$X\sim\exp(1)$: $X-h(X)=X-(1-\alpha)X=\alpha X$

$M_{X-h(X)}(r)=E\left[e^{(r\alpha)X}\right]=\dfrac{1}{1-\alpha r}$

(2) $c_h=?$

$c_h=(1+0.4)\lambda E\left[(1-\alpha)X\right]=1.4\lambda(1-\alpha)$

$\Rightarrow c-c_h=1.25\lambda-1.4\lambda(1-\alpha)=\lambda(1.4\alpha-0.15)$

Solution

Our eq. for R_u is;

$$\boxed{\frac{\alpha r}{1-\alpha r} = (1.4\alpha - 0.15)r}$$

$\psi(u) \le e^{-R_u \cdot u}$ $\Rightarrow$ maximize R w.r.t α (i.e. $\frac{\partial R}{\partial \alpha} = 0$)

$$\alpha r = (1.4\alpha - 0.15)r(1-\alpha r)$$

$$= (1.4\alpha - 0.15)r - \alpha(1.4\alpha - 0.15)r^2 \qquad \div \text{ by } r$$

$$r = \frac{(1.4\alpha - 0.15) - \alpha}{\alpha(1.4\alpha - 0.15)}$$

$\Rightarrow \alpha^* \approx 0.69 \Rightarrow R_u^* \approx 0.22$

MM Ruin
$\Rightarrow$ Retaining 69% of losses

Example 2: Excess of Loss Reinsurance

~~Consider the Cramér-Lundberg process with $X \sim \exp(1)$, $\theta = 0.25$ and $\xi = 0.4$. There is proportional reinsurance with retention α, i.e. $X - h(X) = \alpha X$. What is the retention α that will maximize R_h?~~

$X \sim \exp(1)$, $\theta = 0.25$, $\xi = 0.4$, $h(X) = (X - d)_+$

↳ Solve for d^*

1) $$M_{X-h(X)}(r) = \int_0^\infty e^{r(X - h(x))} e^{-x}\,dx = \int_0^d e^{rx} e^{-x}\,dx + \int_d^\infty e^{rd} e^{-x}\,dx$$

↳ $X - h(X) = X$ ↳ $X - h(X) = d$

$$= \frac{1 - r e^{-d(1-r)}}{1 - r}$$

Solution

$$c_h = 1.4\lambda E[h(X)] = 1.4\lambda \int_d^\infty (x-d)e^{-x}\,dx = 1.4\lambda e^{-d}$$

So the eq for R_h is:

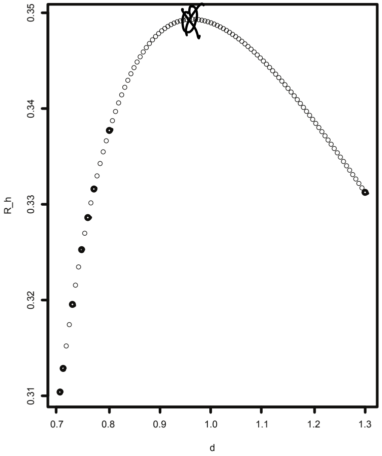
$$1 + (1.25 - 1.4e^{-d})r - \frac{1 - re^{-d(1-r)}}{1-r} = 0$$

Unable to isolate R in terms of d

$\Rightarrow$ Use Solver

Solution

Solution



Use software to maximize R_h, we have

$$d^* = 0.9632226$$

and

$$R_h^* = 0.3493290,$$

which is much higher (better) than the best we could achieve with proportional reinsurance.

(we had $R_h^* \approx 0.22$)

A Theorem

If

- We are in a Cramér-Lundberg setting
- We are considering two reinsurance treaties, one of which is excess of loss
- Both treaties have same expected payments and same premium loadings

then

- The adjustment coefficient with the excess of loss treaty will always be at least as good (high) as with any other type of reinsurance treaty