MATH 4281 Risk Theory–Ruin and Credibility

Summary of Module 2

March 2, 2021

Motivation

Recall the outline of this course

Q1: What do you do when L is equal to a sum of smaller RVs?
$\Rightarrow$ Module 1: Aggregate Loss Models

Q2: How do you introduce **time** to this model?
$\Rightarrow$ Module 2: Ruin Theory

Q3: How do I estimate the parameters of the model for L...if I don't have a nice heterogeneous sample?
$\Rightarrow$ Module 3: Credibility

Recall the beginning of this module

Q1 What happens if we can't pay all the claims?
$\Rightarrow$ Ruin

Q2 How do we set premiums to guarantee that we can?
$\Rightarrow$ We can't 100% eliminate ruin but we can add safety loading to at least make it less than sure .

Q3 How does Time factor in to this?
In models like the Cramér-Lundberg process we can quantify how our premium and (random) loss rates affect ultimate ruin

Stochastic Processes

Stochastic Processes

Randomness + Time = Stochastic Processes

- A *stochastic process* is any collection of random variables $X(t)$, $t \in T$. This stochastic process is denoted as

$$\{X(t), t \in T\}.$$

- In this class we studied 3 kinds of stochastic processes:
  1. Counting Processes (e.g. Poisson)
  2. Compound Poisson Processes (e.g. Aggregate Losses)
  3. The Cramér-Lundberg Process (Cash + Revenue - Aggregate Losses)

Poisson process

A counting process $\{N(t), t \geq 0\}$ is a *Poisson process* with rate λ, for $\lambda > 0$, if:

1. $N(0) = 0$;
2. it has independent increments; and
3. the number of events in any interval of length t has a Poisson distribution with mean λt. That is, for all $s, t \geq 0, n = 0, 1, ...$

$$\Pr[N(t+s) - N(s) = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

Compound Poisson process

We define a Compound Poisson process $\{S(t), t \geq 0\}$ like so:

$$S(t) = \sum_{i=1}^{N(t)} X_i.$$

Where:

- $\{N(t)\}$ is a Poisson process with parameter λ
- $\{X_i\}$ are iid $\sim P(x)$

The Cramér-Lundberg process

Model for the surplus of a non-life insurer at time t:

$$U(t) = \underbrace{u_0 + ct}_{\text{Revenue}} - \underbrace{\sum_{i=1}^{N(t)} X_i}_{\text{Losses}}$$

where

- u_0 initial surplus
- c premium rate:
- $\sum_{i=1}^{N(t)} X_i$ aggregate loss up to time t

The Cramér-Lundberg process

Furthermore if:

- the premium rate is $c = (1+\theta)\lambda E[X]$
- where θ is called the relative security loading.
- and, $\sum_{i=1}^{N(t)} X_i$ is a Compound Poisson (X_i independent of N Poisson)

$\Longrightarrow \{U(t), t \geq 0\}$ is called the Cramér-Lundberg process.

Decision Theory and Ruin

- We spoke about how there are many different ways to quantify decision making.
- We spoke about how utility was developed by economists and ruin theory was developed by actuarial science.
- The key criteria of ruin theory: we want to minimize the probability that the surplus of an insurance company becomes negative!

The probability of ruin

- Recall the Cramér-Lundberg model:

$$U(t) = u_0 + ct - \sum_{i=1}^{N(t)} X_i$$

- The time to ruin T is defined as

$$T = \inf\{t \geq 0 | U(t) < 0\}.$$

- The probability that the company would be ruined by time t is denoted by

$$\psi(u_0, t) = \Pr[T < t].$$

Avoiding Ultimate Ruin

- Finally, the probability of ultimate ruin is

$$\psi(u_0) = \Pr(T < \infty) = \lim_{t \to \infty} \psi(u_0, t) \geq \psi(u, t).$$

- The Net Profit Condition (NPC):

$$c \leq \lambda \mathbb{E}[X_i] \Rightarrow \psi(u_0) = 1$$

- To ensure the NPC holds we add our "safety loading" :

$$c = (1 + \theta)\lambda \mathbb{E}[X]$$

The Lundberg Inequality

How to calculate the probability of ruin

- Usually you cannot do so analytically (with exceptions for exponential and mixtures of exponential losses).
- However the The Lundberg Inequality provides us with a way of approximating the ruin probability such that we can derive useful qualitative results.
- It is a meaningful result assuming moments of the severity exist and we are using the Cramér-Lundberg model.

The adjustment coefficient

In the Cramér-Lundberg model, consider the excess of losses over premiums over the interval $[0, t]$: $S(t) - ct$. We define the adjustment coefficient R as the first positive solution of the following equation in r:

$$M_{S(t)-ct}(r) = E\left[e^{r(S(t)-ct)}\right] = e^{-rct}e^{\lambda t[M_X(r)-1]} = 1,$$

Recall $c = (1+\theta)\lambda E[X]$. So, the adjustment coefficient R is the first positive of the following equation:

$$1 + (1+\theta)rE[X] = M_X(r)$$

The Theorem

1. Let $R > 0$ be the adjustment coefficient. If $\{U(t)\}$ is a Cramér-Lundberg process with $\theta > 0$, then for $u \geq 0$

$$\psi(u) = \frac{e^{-Ru}}{E\left[e^{-RU(T)} \middle| T < \infty\right]}.$$

2. Since $U(T) < 0$, we have then (Lundberg's exponential upper bound)

$$\psi(u) < e^{-Ru}.$$

An example-why is this bound useful?

[1]In some ruin process, the individual claims have a gamma(2, 1) distribution. Determine the loading factor ℓ as a function of the adjustment coefficient R. Also, determine $R(\ell)$. Using a sketch of the graph of the mgf of the claims, discuss the behaviour of R as a function of ℓ .

i.e. Safety loading

i) Find $R(\ell)$ $1+(1+\ell)\cdot 2R = M_X(R)$

Recall $X \sim \Gamma(K, \theta)$ $\Rightarrow M_X(t) = \left(\frac{1}{1-\theta}\right)^K$

$1+(1+\ell)2R = \frac{1}{(1-R)^2}$

$\Rightarrow \ell = \frac{R(3-2R)}{2(1-R)^2}$

[1]Kaas 4.3 #8

An example

ii) $R(\alpha) = ?$

$$\alpha = \frac{R(3-2R)}{2(1-R)^2} = \frac{3R(1-R)+R^2}{2(1-R)^2} = \frac{3}{2}\underbrace{\left(\frac{R}{1-R}\right)}_{=x} + \frac{1}{2}\underbrace{\left(\frac{R}{1-R}\right)^2}_{x^2}$$

$$\Leftrightarrow x^2 + 3x - 2\alpha = 0$$

$$\Rightarrow x = \frac{-3 \pm \sqrt{9+8\alpha}}{2} \Rightarrow R = \frac{3+4\alpha-\sqrt{9+8\alpha}}{4(1+\alpha)}$$

Note: Pick the root as $0 < R < 1$

Optimal Reinsurance

Assumptions

- Let $0 \leq h(x) \leq x$ be the amount paid by the reinsurer for a claim with amount x i.e:
  - $h(X) = (1-\alpha)X$ for proportional reinsurance.
  - $h(X) = (X-d)_+$ for excess of loss reinsurance.

- Reinsurance is non cheap and that the loading on reinsurance premiums is $\xi > \theta > 0$. So the reinsurance premium say c_h is:

$$c_h = (1+\xi)\lambda E[h(X)]$$

Assumptions

- With reinsurance, the Cramér-Lundberg process becomes

$$U(t) = u + (c - c_h)t - \sum_{i=1}^{N(t)} (X_i - h(X_i)).$$

- With reinsurance, the adjustment coefficient, R_h, is then the non-trivial solution to

$$\lambda \left[m_{X-h(X)}(r) - 1 \right] = (c - c_h)r.$$

Equivalently,

$$\lambda + (c - c_h)\, r = \lambda \int_0^\infty e^{r[x-h(x)]} p\,(x)\, dx.$$

A Theorem

If

- We are in a Cramér-Lundberg setting
- We are considering two reinsurance treaties, one of which is excess of loss
- Both treaties have same expected payments and same premium loadings

then

- The adjustment coefficient with the excess of loss treaty will always be at least as good (high) as with any other type of reinsurance treaty