MATH 4281 Risk Theory–Ruin and Credibility

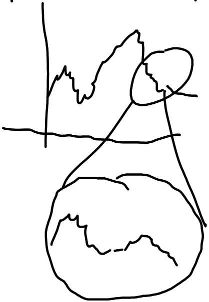
Module 2 Bonus: Some other applications of Ruin Theory

March 2, 2021

Review of the Itô calculus

The building blocks

Nowhere Diff.



Define the Wiener process W_t by:

- $W_0 = 0$
- W_t is continuous
- W_t has independent increments
- $W_t - W_s \sim \mathcal{N}(0, t-s)$

Recall we can recover this as the limit of a random walk as the number of steps goes to infinity.

Itô Integrals

- We can then define integrals with respect to W_t.
- Assume f_t is *adapted to* W_t. Fancy way of saying it shares the probability space.
- Take a partition of $[0, t]$ into n intervals denoted π_n and:

$$\int_0^t f_t \, dW = \lim_{n \to \infty} \sum_{[t_{i-1}, t_i] \in \pi_n} f_{t_{i-1}} (W_{t_i} - W_{t_{i-1}})$$

Itô Processes

- We can then construct *Itô Processes*:

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, d\overset{\omega}{\cancel{B}}_s$$

- Or in differential form:

$$dX_t = \cancel{X_0} + \mu_t dt + \sigma_t d\overset{\omega}{\cancel{B}}_t$$

Doing Calculus with Random Variables

- Given an Itô Process- how does a function of it behave (Real world example: a derivative price as a function of a random stock)

- Assume X_t satisfies $dX_t = \cancel{X_0} + \mu_t dt + \sigma_t dB_t$

- Assume that $f(t, X)$ is $C^2(\mathbb{R})$ then:

$$df(X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} \, dB_t$$

- This is the famous Itô's lemma

Generators

- Let τ be a stopping time (recall from our lectures)
- We have a nice result called Dynkin's formula:

$$\mathbf{E}[f(X_\tau)] = f(X_0) + \mathbf{E}^x\left[\int_0^\tau Af(X_s)\,\mathrm{d}s\right]$$

  Where A is the *generator of* X_t

- In previous slide this would mean:

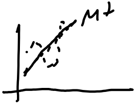
$$A = \frac{\partial}{\partial t} + \mu_t \frac{\partial}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2}{\partial x^2}$$

- There is a deep link between the algebra of differential operators and stochastic processes. Hence why PDEs are common in finance and insurance.

Examples

(Returns)

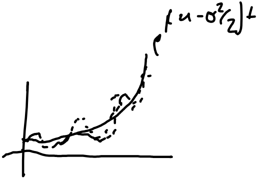
- Brownian Motion with drift:



(Stocks)

$$dX_t = \mu dt + \sigma dW_t$$

- Geometric Brownian Motion:

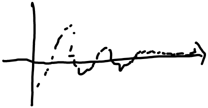


(Interest rates)

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

- Ornstein–Uhlenbeck (Mean Reversion) process:

$$dX_t = -\theta X_t dt + \sigma X_t dW_t$$



Computing Ruin Probabilities

A simple example

- For general processes computing Ruin Probabilities involves the solution to a very complex Partial Integro-Differential Equation (PIDE). But sometimes we can get lucky.
- Consider a simple BM with drift. We start at A_0 and have and *additive* wealth dynamic:

$$dA_t = \mu dt + \sigma dW_t \tag{1}$$

- We want to apply out optimal stopping theorem technique we learned so we will construct a martingale by guessing:

$$M_t = e^{-\alpha(A_t - A_0)} \tag{2}$$

A simple example

1. First we need to guarantee (2) is a martingale. Applying Itô's Lemma:

$$dM_t = \underbrace{\left(-\alpha\mu M_t + \alpha^2 \frac{\sigma^2}{2} M_t\right)}_{=0} dt + (-\alpha\sigma M_t)\, dW_t$$

2. Setting the drift equal to zero gives $\alpha = \frac{2\mu}{\sigma^2}$. This makes M_t a *local martingale* but given the integrability of M_t it is a martingale as well. "Well behaved"

A simple example

3 Define our stopping time as $\tau = \inf\{t | M_t > a \text{ or } M_t < b\}$.

4 Applying the optimal stopping theorem:

$t=0 \qquad t=\tau$

$$E[M_0] = E[M_\tau]$$

$$1 = e^{-\alpha(b-A_0)}P(M_\tau = b) + e^{-\alpha(b-A_0)}(1 - P(M_\tau = b))$$

$$P(M_\tau = b) = \frac{e^{-\alpha A_0} - e^{-\alpha a}}{e^{-\alpha b} - e^{-\alpha a}}$$

5 Send $b \to \infty$ and $a \to 0$ and we have the ruin probability is $1 - e^{-\alpha A_0}$ and the survival probability its $e^{-\alpha A_0}$.

Remarks

- So if we maximize α we minimize ruin!
- Interestingly this can be extended to other processes (using a more complex proof).
- That is minimizing $\frac{\mu_t}{\sigma_t}$ where μ_t and σ_t are the drift and diffusion parts of the generator A minimizes ruin. A very useful result!

Remarks

- Notice this is a similar result to the Lundberg inequality. In fact it is only the discontinuity of the CL process that prevents an exact match.
- Not that surprising: If we have probability distributions that are exponentially bounded (Markov inequality) then for some limit we should see exponentially behaved probabilities.
- What if we don't have this...?

Investing your insurance float

Do what?!

- Often an insurance company will invest it's premiums.
- Famous example of this is Warren Buffet. Buffet actually accessed a smaller cost of capital than other investors by investing his insurance companies surplus or "float".
- There is also something called "convergence capital" where low bond yields are forcing reinsurers to invest their premiums in hedge funds (personally I think this is a bad idea...but no one asked me).

- Consider an investor who invests in a risky asset S_t described by a GBM and a bank account B_t with interest rate r i.e:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{(1)} \quad \text{and} \quad dB_t = rB_t dt$$

- Their wealth X evolves according to the SDE:

$$\begin{aligned}
X_t &= B_t + \gamma S_t \\
dX_t &= rB_t dt + \gamma S_t(\mu dt + \sigma dW_t^{(1)}) \\
dX_t &= \underbrace{B_t}_{(1-f)X_t} rdt + \underbrace{\gamma S_t}_{fX}(\mu dt + \sigma dW_t^{(1)}) \\
\rightarrow dX_t &= \underbrace{X_t[f(\mu - r) + r]}_{``\mu_t"} dt + [\underbrace{X_t f \sigma}_{``\sigma_t"}] dW_t^{(1)}
\end{aligned}$$

- f is the (potentially dynamic) fraction of total wealth X_t invested in the risky asset.

Take our model

- Take the CL model of net claims we studied in class:

$$Y_t = ct - \sum_{i=1}^{N(t)} X_i$$

- Take $a = c - \lambda E[X]$ and $b^2 = \lambda E[X^2]$
- We can approximate Y_t by a BM with drift (more reasonably for some times scales and parameters than others):

$$Y_t \approx adt + bdW_t^{(2)}$$

Putting it together

Q; optimal f?

- If we add the net insurance claims our model for the insurance company now becomes:

$$dX_t = \overbrace{X_t[f(\mu - r) + r]dt + [X_t f\sigma]dW_t^{(1)}}^{\text{Investment income}} + \overbrace{adt + bdW_t^{(2)}}^{\text{net claims}}$$

- From the generator of this process we can now get:

abusing notation

$$\mu_t = X_t[f(\mu - r) + r] + a$$

$$\sigma_t = [X_t f\sigma]^2 + b^2 + 2\rho b[X_t f\sigma]$$

$\rho \rightarrow$ corr of $W^{(1)}$ & $W^{(2)}$

Putting it together

Finding the f that maximizes $\frac{\mu_f}{\sigma_f^2}$ $= \text{"}\alpha\text{"}$ gives:

$$f_\psi = \frac{1}{\mu - r}\left[\sqrt{\left(rx + a - \frac{\rho b(\mu - r)}{\sigma}\right)^2 + (1-\rho^2)b^2\frac{(\mu - r)^2}{\sigma^2}} - (rx + a)\right]$$

"Ruin optimal"

What is Optimal: Kelly vs Ruin vs ?

A few Questions

- Say $b = \rho = 0$ and $a = -c$ i.e. some consumption an investor may need to satisfy. You can show that:

$$f_{\psi}^{*} = \begin{cases} \frac{2|rx-c|}{x\sigma^2}\frac{1}{f_g^*} & rx - c < 0 \\ 0 & rx - c \geq 0 \end{cases}$$

Interest "r" pays for c

- Where f_g^* is the "Kelly" or growth optimal fraction.

$f_g = \frac{\mu - r}{\sigma^2}$

Min t to hit arbitrary wealth

- So a Ruin theoretic approach will be much much more conservative...which is correct?