MATH 4281 Risk Theory–Ruin and Credibility

Module 3: Credibility Theory (cont.)

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The Story so Far

Credibility

- Last class we discussed how actuaries began to use estimates of the kind:

$$\text{Credibility Premium } = z_j \overline{X}_j + (1 - z_j)\,\overline{X}$$

  In situations (like auto policies) where individual rate setting is hard.

- We then spoke about how such estimates can be justified through a *Bayesian* framework. Here we could incorporate past experience and expertise into our model.

Recall the problem

Assume:

- every risk j in the collective is characterized by its individual risk profile $\theta_j \in \Theta$ that does not change over time and that we can't observe.
- Θ may be either qualitative (e.g. good/bad) or quantitative (e.g. average number of accidents per year).
- we have T observations $X_{j1}, \ldots, X_{jT}$

We want to estimate

$$\mu(\theta_j) = E\left[X_{j,T+1} | \theta_j\right]$$

but θ_j is unknown to the insurer...

Bayesian Thinking

- Recall in the Bayesian framework probability represents "belief" in the form of the distribution on θ. Recall Bayes theorem:

$$\pi(\theta|x) = \underbrace{\left[\frac{f(x \mid \theta)}{f(x)}\right]}_{\text{the effect of data/evidence}} \times \pi(\theta)$$

- Given some sample x, we *update* our beliefs about θ and this changes our *Prior* $\pi(\theta)$ into our *Posterior* $\pi(\theta|x)$.

Bayesian Thinking

- This isn't good enough though, we want to score our errors in some way. Some errors are less significant than others.

- So we make use of a loss function. Given an estimator $\hat{\delta}(x)$ for θ we want to minimize:

$$\int_{\Theta} L(\theta, \hat{\delta}(\mathbf{x}))\ \underline{d\Pi(\theta|x)}.$$

posterior

- We showed last class that under a quadratic loss:

$$\widetilde{\delta(\mathbf{x})} = \int \theta d\Pi(\theta|x) = E[\theta|x]$$

The Bayes Premium

The Bayes Premium

- Remember though that we are specifically interested in the *mean* of $X_{j,T+1}$.
- To that end we we introduce the following. Given θ the mean of X is given by $\mu(\theta)$ and we use the loss function:

$$L(\theta, \hat{\mu}(\mathbf{x})) = (\mu(\theta) - \hat{\mu}(\mathbf{x}))^2$$

- From last class we know this will give:

$$P^{Bayes} \equiv \widetilde{\mu(\theta)} = E\left[\mu(\Theta)|x\right] = \int_{\Theta} \mu(\theta)\pi(\theta|x)$$

i.e. the expected mean under the posterior!

Other premiums

This also gives use the notion of the collective premium (a number)

$$P^{coll} = m = \int_{\Theta} \mu(\theta)\, \mathrm{d}\Pi(\theta) = E[X_{j,T+1}]$$

E[E[X|θ]] = ∫ x f(x) dx

i.e. experience-free

N.B:

- without experience/sample $P^{coll} = P^{Bayes}$.
  (Think of $z_j \overline{X}_j + (1 - z_j)\,\overline{X}$)

- The quadratic loss of the collective premium is

$$E\left[(m - \mu(\Theta))^2\right] = \underbrace{E\left[Var(\mu(\Theta)|\mathbf{X})\right]}_{\text{Quad. Loss of } P^{Bayes}} + Var\left(E[\mu(\Theta)|\mathbf{X}]\right)$$

How to calculate P^{Bayes}

Raw materials:

- T realizations $\mathbf{x}$ of X
- the distribution of $\mathbf{X}|\Theta$,

$$F_{X|\Theta}(x|\theta) = \Pr[X \leq x|\Theta = \theta]$$

- the *a priori* distribution of Θ,

$$\Pi(\theta) = \Pr[\Theta \leq \theta]$$

Procedure:

- Determine the *a posteriori* distribution $\pi(x|\theta)$
- Calculate P^{Bayes} with the help of $\pi(x|\theta)$

Example 1:Poisson–gamma

Suppose that given $\Theta = \theta$, past losses $X_1, \cdots, X_T$ are independent and Poisson distributed with Poisson parameter θ which follows a gamma distribution with probability density function

$X_j \sim Poi(\theta_j)$

$$\pi(\theta) = \frac{\beta^\alpha \theta^{\alpha-1} e^{-\theta\beta}}{\Gamma(\alpha)}, \quad \theta > 0.$$

Determine the Bayesian premium. (= mean of $\mu(\theta)$ under posterior)

$$\pi(\theta | x) = \frac{f(x|\theta)\pi(\theta)}{f(x)} \propto f(x|\theta)\pi(\theta)$$

$$f(x|\theta)\,\pi(\theta) \quad \text{(} f(x|\theta) \rightarrow \text{Likelihood)}$$

$$= \left(\prod_{j=1}^{T} f(x_j|\theta) \right) \pi(\theta)$$

$$= \left(\prod_{j=1}^{T} \frac{\theta^{x_j} e^{-\theta}}{x_j!} \right) \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$$

$$z = \exp(\ln(z))$$

$$\prod \overset{\ln}{\longleftrightarrow} \sum$$

$$= \exp\left\{ \sum_{j=1}^{T} \left(x_j \ln(\theta) - \theta - \underbrace{\ln(x_j!)}_{cte} \right) + \underbrace{\alpha \ln\beta}_{cte} - \underbrace{\ln\Gamma(\alpha)} + (\alpha-1)\ln(\theta) - \beta\theta \right\}$$

$$\propto \exp\left\{ \ln(\theta) \left(\sum x_j + \alpha - 1 \right) - (T+\beta)\theta \right\} = \theta^{\left(\sum x_j + \alpha - 1\right)} e^{-(T+\beta)\theta}$$

$$\pi(\theta|x) \propto \theta^{(\sum x_j + \alpha - 1)} e^{-(T+\beta)\theta}$$

proportional to $Gamma(\sum x_j + \alpha, T+\beta)$, where $\sum x_j + \alpha = \tilde{\alpha}$ and $T+\beta = \tilde{\beta}$

proportional to (not alpha)

i.e. $\int \pi(\theta|x) = \int C\, \theta^{\tilde{\alpha}-1} e^{-\tilde{\beta}\theta} d\theta$

$$\Rightarrow C = \left(\int \theta^{\tilde{\alpha}-1} e^{-\tilde{\beta}\theta} d\theta\right)^{-1}$$

$$\Rightarrow \pi(\theta|x) \sim Ga(\tilde{\alpha}, \tilde{\beta})$$

$$P^{Bayes} = \frac{\tilde{\alpha}}{\tilde{\beta}} = \frac{\sum x_j + \alpha}{T+\beta} = \underbrace{\frac{T}{T+\beta}}_{z} \cdot \underbrace{\left(\frac{\sum x_j}{T}\right)}_{= \bar{x}} + \underbrace{\frac{\beta}{T+\beta}}_{1-z} \cdot \underbrace{\left(\frac{\alpha}{\beta}\right)}_{= ?}$$

Note: $P^{Bayes} = Z\bar{X} + (1-Z)\mu$ $\quad Z = \frac{T}{T+\beta}$ $\quad \mu = \alpha/\beta$

Q: Significance of μ?
↳ we would like $\mu = E[X]$, is it?

A: Yes

$$f(x) = \int f(x|\theta)\,\pi(\theta)\,d\theta = \binom{\alpha+x-1}{x}\left(\frac{1}{1+1/\beta}\right)^{\alpha}\left(\frac{1/\beta}{1+1/\beta}\right)^{x}$$

i.e. $X \sim NegBin(\alpha, \beta) \Rightarrow \mu = E[X] = \alpha/\beta$

Example 2:Exponential–gamma

$$f_{X|\Theta}(x|\theta) = \theta e^{-\theta x} \quad \{x > 0, \theta > 0\} \qquad \text{Exponential}(\theta)$$

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta} \quad \{\alpha, \beta > 0, \theta > 0\} \qquad \text{gamma}(\alpha, \beta)$$

P^{Bayes} = mean of $\pi(\theta|x)$

$$\pi(\theta|X) \propto f(X|\theta)\pi(\theta) = \left(\prod_{i=1}^{T} \theta e^{-\theta x_i}\right) \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$$

$$= \exp\left\{ \sum_i [\ln(\theta) - \theta x_i] + (\alpha-1)\ln(\theta) - \beta\theta \right\}$$

$$\propto \theta^{(T+\alpha)} \exp(-\theta(\textstyle\sum x_i + \beta))$$ looks like $Ga(\alpha + T, \beta + \sum x_i)$ ($\tilde{\alpha}$, $\tilde{\beta}$)

So $\pi(\theta|x) \sim Ga(\tilde{\alpha}, \tilde{\beta})$ so $P^{Bayes} = ?$

In Previous Ex

$\pi(\theta|x) = Ga(\tilde{\alpha}, \tilde{\beta})$ $\quad P^{Bayes} = E[\mu(\theta)|X] = E[\underbrace{E[X_{i,T+1}|\theta]}_{=\theta} | X]$

as $X \sim Poi(\theta)$

But Now $\mu(\theta) = 1/\theta$ i.e. mean of an $\exp(\theta)$ dist.

$P^{Bayes} = E[1/\theta \,|\, X]$ i.e. the mean of $1/\theta$ $\quad \theta \sim Ga(\tilde{\alpha}, \tilde{\beta})$

$\Rightarrow$ Mean of $InvGa(\tilde{\alpha}, \tilde{\beta})$

$$= \frac{\tilde{\beta}}{\tilde{\alpha} - 1}$$

$$f(x) = \int_0^\infty f(x|\theta)\,\pi(\theta)\,d\theta = \int \theta e^{-\theta x} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}\,d\theta$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int \theta^\alpha e^{-\theta(\beta+x)}\,d\theta$$

(multiply by 1)

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{(\beta+x)^{\alpha+1}} \int_0^\infty \frac{(\beta+x)^{\alpha+1}}{\Gamma(\alpha+1)} \cdot \theta^\alpha e^{-\theta(\beta+x)}\,d\theta$$

Pareto pdf $\longrightarrow = \frac{\alpha\beta^\alpha}{(\beta+x)^{\alpha+1}}$

$= 1$ (Ga pdf)

$m = E[X] = \frac{\beta}{\alpha-1}$

$$P^{Bayes} = \frac{\tilde{\beta}}{\tilde{\alpha}-1} = z\bar{X} + (1-z)m \quad \text{where} \quad z = \frac{T}{T+\alpha-1}$$

Example 3: Bernoulli-Beta

$$f_{X|\Theta}(x|\theta) = \theta^x(1-\theta)^{1-x} \quad \{x = 0, 1\}\text{Bernouilli}(\theta)$$

$$\pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \quad \{0 < \theta < 1\}\text{Beta}(\alpha,\beta)\ \{\alpha,\beta > 0\}$$

Exercise: Geometric–Beta

$$f_{X|\Theta}(x|\theta) = \theta(1-\theta)^x \quad \{x \in \mathbb{N}\} \qquad \text{Geometric}(\theta)$$

$$\pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \quad \{0<\theta<1\} \quad \text{Beta}(\alpha,\beta)\ \{\alpha,\beta>0\}$$

$$f_X(x) = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+1)\Gamma(\beta+x)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+x+1)} \quad \{x \in \mathbb{N}\}$$

$$\mu(\theta) = \frac{1-\theta}{\theta} \quad \text{and} \quad m = \frac{\beta}{\alpha-1}.$$

$\pi_{\mathbf{x}}(\theta)$ is Beta$(\widetilde{\alpha}, \widetilde{\beta})$ with

$$\widetilde{\alpha} = \alpha + T \quad \text{and} \quad \widetilde{\beta} = \beta + S.$$

Thus,

$$P^{Bayes} = \frac{\widetilde{\beta}}{\widetilde{\alpha}-1} = \frac{\beta+S}{\alpha+T-1} = z\bar{X} + (1-z)m \quad \text{with} \quad z = \frac{T}{T+\alpha-1}.$$

Exercise: Normal–Normal

$$f_{X|\Theta}(x|\theta) = \phi\left(\frac{x-\theta}{\sigma_2}\right) \quad \{-\infty < x, \theta < +\infty, \sigma_2 > 0\} \quad \text{Normal}(\theta, \sigma_2^2)$$

$$\pi(\theta) = \phi\left(\frac{\theta-m}{\sigma_1}\right) \quad \{-\infty < \theta, m < +\infty, \sigma_1 > 0\} \quad \text{Normal}(m, \sigma_1^2)$$

$$f_X(x) = \phi\left(\frac{x-m}{\sqrt{\sigma_1^2+\sigma_2^2}}\right) \quad \{-\infty < x < +\infty\} \quad \text{Normal}(m, \sigma_1^2 + \sigma_2^2)$$

$$\mu(\theta) = \theta \quad \text{and} \quad m = m.$$

$\pi_{\mathbf{x}}(\theta)$ is Normal$(\widetilde{m}, \widetilde{\sigma}_1^2)$ with

$$\widetilde{m} = \frac{\sigma_1^2 S + \sigma_2^2 m}{T\sigma_1^2 + \sigma_2^2} \quad \text{and} \quad \widetilde{\sigma}_1^2 = \frac{\sigma_1^2\sigma_2^2}{T\sigma_1^2 + \sigma_2^2}.$$

Thus,

$$P^{Bayes} = \widetilde{m} = \frac{\sigma_1^2 S + \sigma_2^2 m}{T\sigma_1^2 + \sigma_2^2} = z\bar{X} + (1-z)m \quad \text{with} \quad z = \frac{T}{T + \sigma_2^2/\sigma_1^2}.$$

A General Model

A useful result

For which pairs $f_{X|\Theta}(x|\theta)$ and $\pi(\theta)$ is $P^{Bayes} = \widetilde{\mu(\Theta)}$ linear?
Equivalently, when is $\widetilde{\mu(\Theta)}$ of the form

$$\widetilde{\mu(\Theta)} = z\bar{X} + (1-z)m\ ?$$

- It is the case for about half a dozen famous examples.
- Jewell (1974) unified these examples
- Gerber (1995) proposed an alternative formulation

A general model

Suppose

- $$f_{X|\Theta}(x|\theta) = \frac{a(x) \cdot b(\theta)^x}{c(\theta)}, \quad x \in A$$

  where

  $$c(\theta) = \int_A a(x) \cdot b(\theta)^x \mathrm{d}x,$$

  and

- $$\pi(\theta) = \frac{c(\theta)^{-m_0} \cdot b(\theta)^{x_0} \cdot b'(\theta)}{d(m_0, x_0)},$$

  where

  $$d(m_0, x_0) = \int c(\theta)^{-m_0} \cdot b(\theta)^{x_0} \cdot b'(\theta) \mathrm{d}\theta.$$

A general model

Then

- $\pi_{\mathbf{x}}(\theta)$ is in the same family of $\pi(\theta)$, with the following updated parameter values (for m_0 and x_0):

$$m_0 + T \quad \text{and} \quad x_0 + \sum_{j=1}^{T} X_j$$

- and finally,

$$P^{Bayes} = \widetilde{\mu(\Theta)} = E[\Theta|X] = \frac{x_0 + S + 1}{m_0 + T} = z\bar{X} + (1 - z)m$$

with

$$z = \frac{T}{m_0 + T}.$$