# MATH 4281 Risk Theory–Ruin and Credibility

Module 3: Credibility Theory finale and the last lecture!

March 25th, 2021

# Non-Parametric Bühlmann model

# Recall from last class

- $X_{jt}$: claims size of policy $j$ during year $t$.
- Available data, $1 \leq j \leq J$, $1 \leq t \leq T$:

| year $t$ | 1 | 2 | 3 | $\cdots$ | $T$ | Risk | Mean |
|---|---|---|---|---|---|---|---|
| policy $j=1$ | $X_{11}$ | $X_{12}$ | $X_{13}$ | $\cdots$ | $X_{1T}$ | $\theta_1$ | $\bar{X}_{1\Sigma}$ |
| policy $j=2$ | $X_{21}$ | $X_{22}$ | $X_{23}$ | $\cdots$ | $X_{2T}$ | $\theta_2$ | $\bar{X}_{2\Sigma}$ |
| policy $j=3$ | $X_{31}$ | $X_{32}$ | $X_{33}$ | $\cdots$ | $X_{3T}$ | $\theta_3$ | $\bar{X}_{3\Sigma}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| policy $j=J$ | $X_{J1}$ | $X_{J2}$ | $X_{J3}$ | $\cdots$ | $X_{JT}$ | $\theta_J$ | $\bar{X}_{J\Sigma}$ |

- $X_{11}, X_{12}, \ldots, X_{JT}$ are *iid* conditional on $\Theta$.
- $\mu(\theta_j) = E[X_{jt}|\Theta = \theta_j]$
- $\sigma^2(\theta_j) = Var(X_{jt}|\Theta = \theta_j)$

$$P^{cred}_{j,T+1} = z\bar{X}_{j\Sigma} + (1-z)m, \quad i = 1, \ldots, J \qquad (1)$$

# Nonparametric estimation (unbiaised estimators)

Estimation of $E[\mu(\Theta)] = m$:

$$\bar{X}_{\Sigma\Sigma} = \frac{1}{J}\sum_{j=1}^{J}\bar{X}_{j\Sigma} = \frac{\sum_{j=1}^{J}\sum_{t=1}^{T}X_{jt}}{JT} \tag{2}$$

Estimation of $E[\sigma^2(\Theta)] = s^2$:

$$\hat{s}^2 = \frac{1}{J}\sum_{j=1}^{J}\hat{s}_j^2 = \underbrace{\frac{1}{J}\sum_{j=1}^{J}\underbrace{\left[\sum_{t=1}^{T}\frac{(X_{jt}-\bar{X}_{j\Sigma})^2}{T-1}\right]}_{\hat{s}_j}}_{\bar{\hat{s}}_j} \tag{3}$$

# Nonparametric estimation (unbiaised estimators)

Estimation of $Var(\mu(\Theta)) = a$:
(Bühlmann's estimator)

$$\hat{a}_B = Max\left\{ \overbrace{\frac{\sum_{j=1}^{J}(\bar{X}_{j\Sigma} - \bar{X}_{\Sigma\Sigma})^2}{J-1} - \frac{1}{T}\hat{s}^2}^{\text{Law of tot. var}} \; ; \; \underbrace{0}_{\because \, Var \geq 0} \right\} \tag{4}$$

(CAS's estimator)

$$\hat{a}_{CAS} = Max\left\{ \frac{\sum_{j=1}^{J}\sum_{t=1}^{T}(X_{jt} - \bar{X}_{\Sigma\Sigma})^2}{JT-1} - \hat{s}^2 \; ; \; 0 \right\} \tag{5}$$

- If $\hat{a} = 0$ then $z = 0$, which makes sense (all risks have the same parameter)

# Example 2

You are given the following past claims data on a portfolio of three classes of policyholders:

| | Year | | |
|---|---|---|---|
| Class | 1 | 2 | 3 |
| 1 | 700 | 800 | 600 |
| 2 | 625 | 500 | 675 |
| 3 | 800 | 850 | 750 |

Estimate the Bühlmann credibility premium to be charged in year 4 for each class of policyholder.

| Class | Y1 | Y2 | Y3 | $\bar{X}_{j\Sigma}$ | $\hat{S}_j^2$ |
|---|---|---|---|---|---|
| 1 | 700 | 800 | 600 | 700 | 10000 |
| 2 | 625 | 500 | 675 | 600 | 8125 |
| 3 | 800 | 850 | 755 | 800 | 2500 |

$X_{\Sigma\Sigma} = 700$ $\quad$ $S^2 = 6875$

1) Find $\bar{X}_{j\Sigma}$'s e.g. $X_{1\Sigma} = \frac{700+800+600}{3} = 700$

2) find $\hat{S}_j^2$ e.g. $\hat{S}_1^2 = \frac{1}{3-1}\left[(700-700)^2+(800-700)^2+(600-700)^2\right]$

$= 10{,}000$

3) Collective stats

4) Find $a$ ( "Var$(\mu(\theta))$" ) & $\hat{k}$

$$a_B = \max\left\{ \frac{1}{3-1}\left[ \underbrace{(700-700)^2 + (600-700)^2 + (800-700)^2}_{\sum_i (\bar{X}_{ji} - \bar{X}_{\Sigma\Sigma})^2} - \underbrace{\frac{6875}{3}}_{\hat{s}^2/T}, 0 \right\} \right.$$

$$= 7708.3$$

$$\hat{k} = \frac{\hat{s}^2}{a_B} \approx 0.9$$

5) We have, $\hat{k}$ and know $T=3$

$$\Rightarrow Z = \frac{T}{T+k} = \frac{3}{3+0.9} \approx 0.8$$

6) Calc the Bühlmann Premiums

$$P_1^{cred} = Z\bar{X}_{1\cdot} + (1-Z)\hat{\mu}$$

$$= (0.8)700 + (0.2)700 = 700$$

$$P_2^{cred} = (0.8)600 + (0.2)700 = 620$$

# The Bühlmann-Straub model

# Adding more realism to the Bühlmann model

- Often we have somewhat coarse data available to us (this is changing but in 1970 when this model was introduced it was even more true).
- Many lines of business have a premium of the type "volume measure" times "premium rate".
- In this case we use an extension of Bühlmann model: Bühlmann-Straub.
- by far the most used and the most important credibility model for insurance practice.

# The model

There are $1 \leq j \leq J$ classes of risk (or contracts).
For the $j$-th class/contract:

- $S_{jt}$ is the aggregate claim amount in year $t$ ( $1 \leq t \leq T$)
- $w_{jt}$ is the "volume" associated to $S_{jt}$ in year $t$
- $X_{jt} = S_{jt}/w_{jt}$ is the claim amount per unit of volume in year $t$
- One (my favourite) interpretation: average claim costs per year at risk in year $t$ if $w_{jt}$ is the number of years at risk during year $t$. That is if:

policies on an individual

$$\underbrace{S_{jt} = \sum_{k=1}^{w_{jt}} Y_{jt,k}}_{\text{aggregate}}$$

and $X_{ij} \to \underline{\text{average}}$

# Assumptions

- risk class/contract $j$ is characterized by its specific risk parameter $\theta_j$, which is the realization of a rv $\Theta_i$
- Conditional on $\Theta_i$, the $\{X_{jt} : t = 1, 2, \cdots, T\}$ are iid with $\mu(\theta_j) = E[X_{jt}|\Theta = \theta_j]$ but now:[1]

$$Var(X_{jt}|\Theta = \theta_j) = \frac{\sigma^2(\theta_j)}{w_{jt}}$$

- the pairs $(\Theta_1, \mathbf{X_1}), (\Theta_2, \mathbf{X_2}), \ldots$ are independent
- $\Theta_1, \Theta_2, \ldots$ are iid (from the structural distribution)

[1] In the previous interpretation, $\mu(\theta_j) = E[Y_j]$ and $\sigma^2(\theta_j) = Var(Y_j)$.

# "New Quantities"

Risk $j$:

- individual risk premium
  $\mu(\theta_j) = E[X_{jt}|\Theta = \theta_j]$
- variance within individ. risk
  $\sigma^2(\theta_j) = w_{jt} Var(X_{jt}|\Theta = \theta_j)$
- aggregate volume
  $w_{j\Sigma} = \sum_{t=1}^{T} w_{jt}$
- weighted mean of outcomes
  $\bar{X}_{j\Sigma} = \sum_{t=1}^{T} \frac{w_{jt}}{w_{j\Sigma}} X_{jt}$

Collective:

- collective premium
  $m = E[\mu(\Theta)]$
- variance between individual risk premiums
  $a = Var(\mu(\Theta))$
- average variance within individual risks
  $s^2 = E[\sigma^2(\Theta)]$
- aggregate volume
  $w_{\Sigma\Sigma} = \sum_{j=1}^{J} w_{j\Sigma}$
- weighted mean of outcomes
  $\bar{X}_{\Sigma\Sigma} = \sum_{j=1}^{J} \frac{w_{j\Sigma}}{w_{\Sigma\Sigma}} \bar{X}_{j\Sigma}$

# If $m$, $s^2$ and $a$ are known

The credibility estimator in the Bühlmann-Straub model is given by

$$P^{cred}_{j,T+1} = z_j \bar{X}_{j\Sigma} + (1 - z_j)m = m + z_j(\bar{X}_{j\Sigma} - m),$$

where

$$\begin{aligned} z_j &= \frac{w_{j\Sigma}}{w_{j\Sigma} + K} \\ K &= \frac{E[\sigma^2(\Theta)]}{Var(\mu(\Theta))} \Bigg\} z\ ? \end{aligned}$$

Remarks:

- the credibility factor $z_j$ now depends on $j$
- if $w_{jt} = 1$, then $w_{j\Sigma} = T$ and $z_j$ is equivalent to the $z$ of the simple Bühlmann model

# If $s^2$ and $a$ are known but $m$ has to be estimated

$$P^{cred}_{j,T+1} = z_j \bar{X}_{j\Sigma} + (1 - z_j)\widehat{m} = \widehat{m} + z_j(\bar{X}_{j\Sigma} - \widehat{m}),$$

where

$$\widehat{m} = \sum_{j=1}^{J} \frac{z_j}{z_\Sigma} \bar{X}_{j\Sigma}, \quad z_\Sigma = \sum_{j=1}^{J} z_j$$

Remarks:

- it can be shown that $\widehat{m}$ is a better estimator of $m$ than $\bar{X}_{\Sigma\Sigma}$
- Quadratic loss:

$$E[(\widehat{m} + z_j(\bar{X}_{j\Sigma} - \widehat{m}) - \mu(\theta_j))^2] = a(1 - z_j)\left(1 + \frac{1 - z_j}{z_\Sigma}\right)$$

- This makes sense- in an non-i.i.d sample, the weighted average where the weights are inversely proportional to the variances is BLUE.

# If $m$, $s^2$ and $a$ have to be estimated

$$P^{cred}_{j,T+1} = \widehat{z}_j \bar{X}_{j\Sigma} + (1-\widehat{z}_j)\widehat{m} = \widehat{m} + \widehat{z}_j(\bar{X}_{j\Sigma} - \widehat{m}),$$

Where we use the following unbiased (weighted) sample statistics:

$E[\sigma^2(\theta_j)]$

$= w_j \mathrm{Var}(X_j)$

$$\widehat{z}_j = \frac{w_{j\Sigma}}{w_{j\Sigma} + \frac{\hat{s}^2}{\hat{a}}}$$

$$\begin{cases} \hat{s}^2 = \frac{1}{J}\sum_{j=1}^{J} \hat{s}_j^2 = \frac{1}{J}\sum_{j=1}^{J}\left(\frac{1}{T-1}\sum_{t=1}^{T} w_{jt}(X_{jt} - \bar{X}_{j\Sigma})^2\right) \\ \hat{a} = \frac{w_{\Sigma\Sigma}}{w_{\Sigma\Sigma}^2 - \sum_{j=1}^{J} w_{j\Sigma}^2}\left\{\sum_{j=1}^{J} w_{j\Sigma}(\bar{X}_{j\Sigma} - \bar{X}_{\Sigma\Sigma})^2 - (J-1)\hat{s}^2\right\} \end{cases}$$

weighted var

# Numerical Example

Past claims data on a portfolio of two groups of policyholders are given below:

| | Group | | Year 1 | Year 2 | Year 3 | Year 4 |
|---|---|---|---|---|---|---|
| Total Claim Amount | 1 | $S_{jt}$ | 8000 | 11,000 | 15,000 | — |
| Number in Group | | $w_{jt}$ | 40 | 50 | 70 | 75 |
| Total Claim Amount | 2 | | 20,000 | 24,000 | 19,000 | — |
| Number in Group | | | 100 | 120 | 115 | 95 |

Estimate the Bühlmann-Straub credibility premium to be charged in year 4 for each group of policyholder.

| | $Y_1$ | $Y_2$ | $Y_3$ |
|---|---|---|---|
| $X_{1t}$ | 200 | 220 | 214.49 |
| $X_{2t}$ | 200 | 200 | 165.22 |

$\frac{8000}{200} = 40$

$w_{1\Sigma} = 40 + 50 + 70 = 160$

$w_{2\Sigma} = 335.00$

| | $Y_1$ | $Y_2$ | $Y_3$ | $\bar{X}_{j\xi}$ | $\hat{S}_j$ |
|---|---|---|---|---|---|
| $X_{1t}$ | 200 | 220 | 214.49 | 212.50 | 4642.86 |
| $X_{2t}$ | 200 | 200 | 165.22 | 188.06 | 45,684.62 |

$X_{\xi\xi} = 195.96$ $\quad \hat{s}^2 = 25,163.74$

① Individual weighted sample mean

$$\bar{X}_{1\xi} = \frac{w_1 X_{11} + w_2 X_{12} + w_3 X_{13}}{w_{1\xi}} = \frac{40 \cdot 200 + 50 \cdot 220 + 70 \cdot 214.29}{160} = 212.50$$

② Individual weighted sample var

$$\hat{S}_1^2 = \frac{1}{3-1}\left[40(200-212.5)^2 + 50(220-212.5)^2 + 70(214.29-212.5)^2\right]$$

$= 4642.86$

③ Collective;

$$\frac{160 \times 212.5 + 335 \times 188.06}{160+335}$$

(4) Find $a$ & $K$

$$a = \frac{(160+335)}{(160+335)^2 - 160^2 - 335^2}\left\{ 160(212.5-195)^2 + 335(188-195)^2 - (2-1)\cdot 25163.74 \right\}$$

$$\approx 182.47$$

$$K = \frac{\hat{s}^2}{a} \approx 137.91$$

6) Find $z$, $\mu$

i) $z_1 = \dfrac{w_{1\Sigma}}{w_{1\Sigma} + K} = \dfrac{160}{160 + 137.91} \approx 0.54$

$z_2 = \dfrac{w_{2\Sigma}}{w_{2\Sigma} + K} = \dfrac{335}{335 + 137.91} = 0.71$

ii) $\hat{\mu} = \sum \dfrac{z_j}{z_\Sigma} \bar{X}_{j\Sigma}$

$= \dfrac{0.54}{0.54 + 0.71}(212.5) + \dfrac{0.71}{0.54+0.71}(188.06)$

$= 198.60$

7)

$P_1^{cred} = w_{14}\left[ z_1 \bar{X}_{1\Sigma} + (1 - z_1)\hat{\mu} \right]$

$= 75\left[0.54(212.5) + (0.46)(198.60)\right] = 15454.87$

$P_2^{cred} = 18158.10$